FEYNMAN PROPAGATOR AND LIMITING ABSORPTION PRINCIPLE ON ASYMPTOTICALLY MINKOWSKI SPACETIMES

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Propagators are one of the fundamental objects in Quantum Field Theory (QFT). Mathematically, it is defined by fundamental solutions to the Klein-Gordon equation such as those given by the advanced/retarded propagators and the Feynman/anti-Feynman propagators. The existence of the advanced/retarded propagators is classically known. These are constructed by solving a Cauchy problem and defined for globally hyperbolic spacetimes. On the other hand, how the Feynman/anti-Feynman propagators are generalized is non-trivial and these existence problem is global in nature and is more complicated. See the introductions of [1], [2], [3], [5], [6] and [7].

The massive Feynman propagator ($m > 0$) in exact Minkowski spacetime is given by

$$(\partial_t^2 - \Delta_y + m^2 - i0)^{-1}. \tag{1}$$

More precisely, its integral kernel (Green function) $(\partial_t^2 - \Delta_y + m^2 - i0)^{-1}(x, x')$ is given by

$$(\partial_t^2 - \Delta_y + m^2 - i0)^{-1}(x, x') = \lim_{\varepsilon \to 0,\ \varepsilon > 0} \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} \frac{e^{i(x-x')\cdot\xi}}{-\tau^2 + |\eta|^2 + m^2 - i\varepsilon} d\xi,$$

where we write $\xi = (\tau, \eta) \in \mathbb{R} \times \mathbb{R}^n$. This formula is defined by using the Fourier analysis and the distribution theory, so how Feynman propagators are defined in more general curved spacetimes is non-trivial from this formula.

Duistermaat-Hörmander [4] gave a precise definition of Feynman propagator in terms of the wavefront set and showed the existence of Feynman parametrices (inverses up to smoothing errors) under general settings. Recently, actual inverses (Feynman propagators) have been constructed on various spacetimes by using the scattering theory or the microlocal theory ([1], [2], [3], [5], [6], [7] and [11]). However, their constructions differ from each other and relationship between such Feynman propagators and the formula (1) had not been addressed. In this study, following a program of [1], [3], we show that the (anti-)Feynman propagator constructed in [6] and [7] coincides with a limit of resolvent as (1) on curved spacetimes which are close to the Minkowski spacetime near spacetime infinity.

Now we state our main theorem. Let g_0 be the Minkowski metric on $\mathbb{R}_x^{n+1} = \mathbb{R}_t \times \mathbb{R}_y^n$ and g_0^{-1} be its dual metric:

$$g_0 = -dx_1^2 + dx_2^2 + ... + dx_{1+n}^2, \quad g_0^{-1} = -\partial_{x_1}^2 + \partial_{x_2}^2 + ... + \partial_{x_{1+n}}^2 = (g_0^{ij})_{i,j=1}^n.$$

We write $\langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}$ and introduce the function space

$$S^k(\mathbb{R}^{n+1}) := \{a \in C^\infty(\mathbb{R}^{n+1}) \mid |\partial_x^\alpha a(x)| \le C\langle x\rangle^{k-|\alpha|}\}, \qquad k \in \mathbb{R}.$$

Assumption 1. *A Lorentzian metric g on $\mathbb{R}^{n+1}$ satisfies the following conditions: The inverse matrix $g^{-1}(x) = (g^{jk}(x))_{j,k=1}^n$ of $g(x)$ satisfies $g^{jk} - g_0^{jk} \in S^{-\mu}(\mathbb{R}^{n+1})$ for some $\mu > 0$.*

Assumption 2 (Null non-trapping condition)**.** *All non-constant null geodesics escape to the spacetime infinity.*

Assumption 3. *There exists a time function $\tilde{t}$ such that $\tilde{t} - t \in C_c^\infty(\mathbb{R}^{n+1})$.*

It is shown in [8], [9] and [11] that $P := -\Box_g$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^{n+1})$ under Assumptions 1 and 2. We denote the unique self-adjoint extension by the same symbol P. Moreover, we set $L^{2,s}(\mathbb{R}^{n+1}) := \langle x\rangle^{-s} L^2(\mathbb{R}^n)$.

Theorem 1. [10] *We suppose Assumptions 1 and 2.*
(i) Let $s > \frac{1}{2}$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Then the limits

$$(P - \lambda \mp i0)^{-1} := \lim_{\varepsilon \to 0,\, \varepsilon > 0} (P - \lambda \mp i\varepsilon)^{-1}$$

exist in $B(L^{2,s}(\mathbb{R}^{n+1}), L^{2,-s}(\mathbb{R}^{n+1}))$.
(ii) In addition, we suppose Assumption 3, $\mu > 1$ and $\lambda = m^2 > 0$. The operator $(P+m^2-i0)^{-1}$ coincides with the (anti-)Feynman propagator defined in [6] *and* [7]*. In particular, $(P+m^2-i0)^{-1}$ is actually a Feynman propagator in the definition of Duistermaat-Hörmander* [4].

Remark 1. *The convention of the Feynman/anti-Feynman propagators in* [6] *and* [7] *are different from ones in physics books. Here we follow the convention in physics books.*

Remark 2. *A weaker statement of (i) is also proved in* [11] *by a different method but under a very short-range condition.*

The proof of the part (i) is essentially due to the Mourre theory, which is commonly used in the scattering theory. A main difficulty here is due to the lack of ellipticity of $P = -\Box_g$.

To describe the key idea of the proof of the part (ii), we rewrite (1) as

$$(\partial_t^2 - \Delta_y + m^2 - i0)^{-1} f(t,y) = \frac{i}{2\sqrt{-\Delta_y + m^2}} \int_{\mathbb{R}} e^{-i|t-t'|\sqrt{-\Delta_y+m^2}} f(t',y)dt$$

for $(t,y) \in \mathbb{R} \times \mathbb{R}^n$. If $f \in C_c^\infty(\mathbb{R}^{n+1})$, then we write

$$(\partial_t^2 - \Delta_y + m^2 - i0)^{-1} f(t,y) = \begin{cases} e^{-it\sqrt{-\Delta_y+m^2}} g_+(t,y) & t >> 1 \\ e^{it\sqrt{-\Delta_y+m^2}} g_-(t,y) & t << -1 \end{cases}$$

for some function $g_\pm$ and it satisfies $(D_t \pm \sqrt{-\Delta_y + m^2})u = 0$ for $\pm t >> 1$. We regard this equation as an analogue of Sommerfeld's radiation condition $(\partial_r - i\sqrt{\lambda})u = o(|x|^{-\frac{n-1}{2}})$ for the Helmholtz equation $(-\Delta - \lambda)u = 0$ on $\mathbb{R}^n$. This observation is more or less justified for more general setting by using the microlocal analysis. Moreover, uniqueness of solutions under the radiation condition yields an identification of Feynman propagators (proof of the part (ii)).

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